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Attracting sets in a deterministic discrete traffic model

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Abstract
The fundamental diagram of the Nagel–Schreckenberg traffic model is derived analytically for the deterministic case using methods and concepts from nonlinear dynamics. It is shown that the possible states of the long-term behaviour form a globally attractive subset which can be well characterized. This attractive set of states is composed of coexisting attractors. The attractor concept is applied to a slow-to-start extension of the model. For this example it is shown that the attractive set consists of coexisting attractors with different macroscopic properties, that can be determined analytically.

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1. Introduction

In recent years traffic models discretized in both space and time have been the object of investigations aiming at determining the influence of various parameters (such as vehicle density, speed limits etc) on the overall behaviour of a traffic system. Another common objective is the possibility of real-time prediction of traffic jams [1]. The model introduced by Nagel and Schreckenberg [2] has been used as a paradigm for many investigations because of its structural simplicity.

For most of these investigations concepts and methods from statistical mechanics have been used in order to characterize the overall speed or flux behaviour of the model with respect to different car densities.

In the following, we intend to present an alternative view using concepts and methods of deterministic nonlinear dynamics. Although the model has much of the character of a many-particle system, we show that statistical methods are not necessary to obtain an exact term for the fundamental diagram. This can be of interest for investigations of related models where averaging over various initial conditions tends to obscure important aspects of the system behaviour. An example for such a model is presented in section 6 and then treated with the new approach.

In section 2 the deterministic case of the Nagel–Schreckenberg model is formulated in a new closed representation. In section 3 the transient and long-term behaviour of this case is investigated; in section 4 we derive an upper bound for the fundamental diagram analytically. In section 5 we sketch a proof that this upper bound is identical to the speed always reached in the long-term behaviour. This proof is based on concepts of nonlinear dynamics. Finally, as already mentioned, a slow-to-start extension, its impact on the model behaviour and the applicability of the previously developed concepts are the focus of section 6.

2. The Nagel–Schreckenberg model

2.1. The verbal formulation by Nagel and Schreckenberg

The Nagel–Schreckenberg model [2] describes the motion of N vehicles in a set of discrete positions ξ with discrete velocities ν on a closed loop of length L by four rules:

(i) *Acceleration.* If the velocity ν_τ of a vehicle at a time step τ is less than the maximal velocity $\nu_{\max}$ and the distance to the next vehicle ahead is larger than $\nu_\tau + 1$, the speed is advanced by one ($\nu_{\tau+1} = \nu_\tau + 1$).
(ii) *Slowing down due to other cars.* If at time step τ a vehicle at position ξ_τ 'sees' another vehicle at position $\xi_\tau + \Delta\xi_\tau$, it reduces its speed to $\Delta\xi_\tau - 1$ in order to avoid a collision ($\nu_{\tau+1} = \Delta\xi_\tau - 1$).
(iii) *Randomized slowing down.* With a given probability p the speed of a vehicle is reduced by one unit, if it was greater than zero ($\nu_{\tau+1} = \nu_\tau - 1$).
(iv) *Car motion.* Each vehicle is advanced ν sites ($\xi_{\tau+1} = \xi_\tau + \nu_\tau$).

In the following investigations we will exclusively consider the deterministic case, where the probability of randomized slowing down is $p = 0$.

2.2. A closed representation of the deterministic case

If we indicate the number of the vehicle by an upper index ($i \in \{1 \ldots N\}$), the deterministic case can be expressed in a rather compact way as follows:

$$\text{Substep 1:}\quad \nu_{\tau+1}^{(i)} = \min(\nu_{\max},\ \nu_\tau^{(i)} + 1,\ \Delta\xi_\tau^{(i)} - 1) \tag{1}$$

$$\text{Substep 2:}\quad \xi_{\tau+1}^{(i)} = \xi_\tau^{(i)} + \nu_{\tau+1}^{(i)}. \tag{2}$$

Substep 1 is a summary of rules (1) and (2). Using this formulation, we easily recognize at first sight the three limiting values for the speed of a vehicle at time $\tau + 1$:

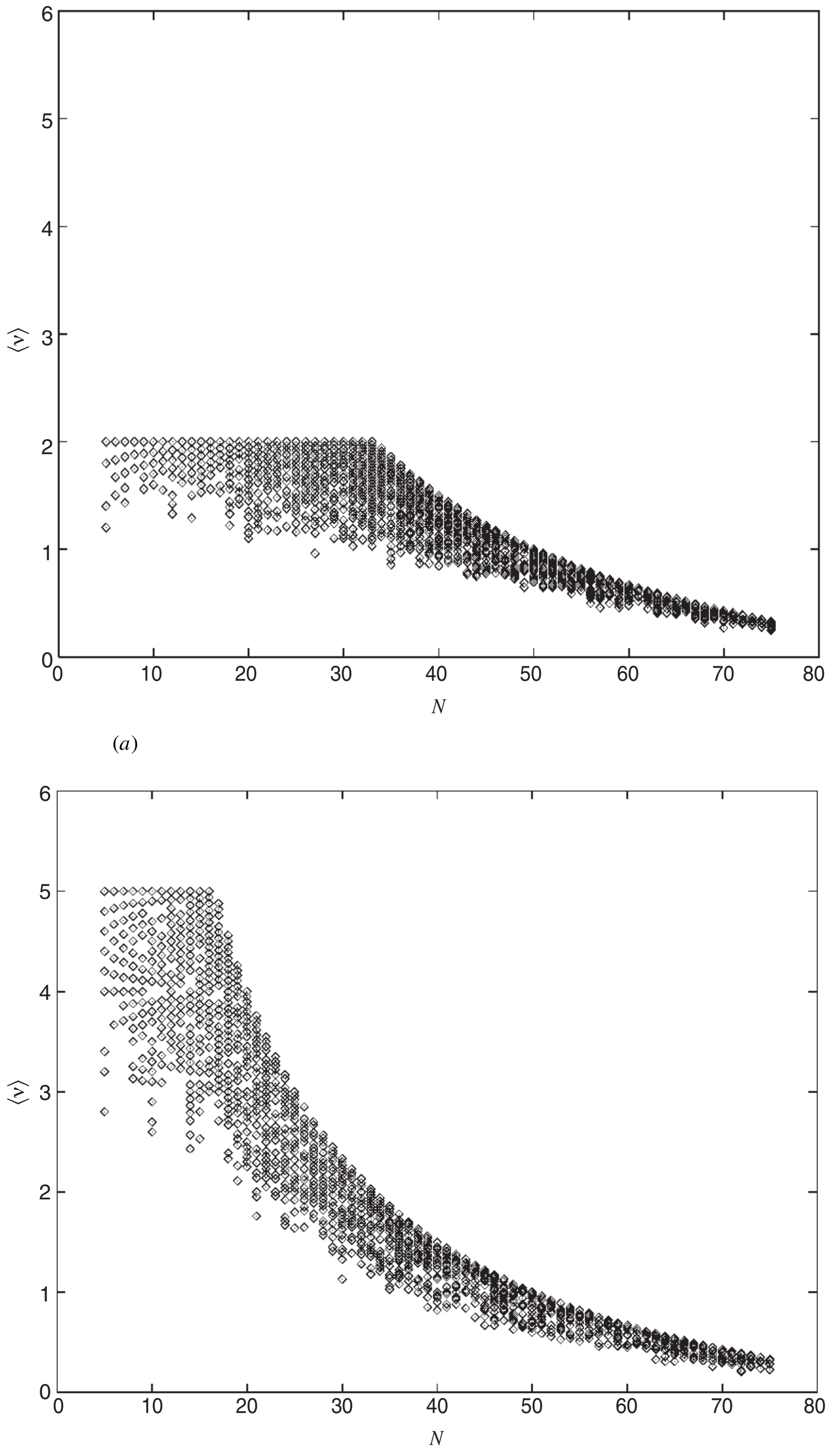
(i) the maximum discrete velocity $\nu_{\max}$;
(ii) the maximum velocity $\nu_\tau^{(i)} + 1$ attainable with maximum acceleration and
(iii) the maximum velocity $\Delta\xi_\tau^{(i)} - 1$ compatible with the free space in front of the vehicle.

Substep 2 is equivalent to rule (4). The explicit formulation as an iterated map for the variables $\xi_\tau^{(i)}$ and $\nu_\tau^{(i)}$ will be helpful for the following investigations.

3. Behaviour of the deterministic Nagel–Schreckenberg model in computer simulations

3.1. Fundamental scatter plots of the Nagel–Schreckenberg model with different discrete maximum velocities $\nu_{\max}$

For given values of the maximum velocity $\nu_{\max}$ and of the length L of the periodic lane and different numbers of vehicles N, the average velocity of the N vehicles at time τ can be



(*a*)

(*b*)

Figure 1. Fundamental scatter plots for $L = 100$ with $N = 10, 11, 12, \ldots, 75$, $\tau = 0, 1, 2, \ldots, 100$: (*a*) $v_{\max} = 2$; (*b*) $v_{\max} = 5$.

computed from

$$\langle v \rangle_\tau := \frac{1}{N} \sum_{i=1}^{N} v_\tau^{(i)}. \tag{3}$$

In this paper we will call a plot of $\langle v \rangle_\tau$ against N for constant L and $v_{\max}$ a fundamental scatter plot. These plots are very similar to the commonly used fundamental diagrams; however, in the scatter plots the values for $\langle v \rangle_\tau$ are plotted for various initial conditions and each τ separately while in the fundamental diagrams the values are averaged over various initial conditions and a large number of τ values (leaving out small values of τ in order to suppress the influence of transient behaviour).

The scatter plots in figure 1 are generated by plotting the values of $\langle v \rangle_\tau$ for $\tau = 0, 1, 2, \ldots, 100$ and different values of N.

Different values of $\langle v \rangle_\tau$ corresponding to different values of τ occur at the same N. The maxima, which will be called 'fundamental maxima', appear to lie on a characteristic curve.

3.2. Transient versus long-term behaviour

If one considers the fundamental scatter plots of $\langle v \rangle_\tau$ for large τ only, however, the values always seem to correspond to the fundamental maxima. This suggests that all lower values result from transient behaviour. Figure 2 shows $\langle v \rangle_\tau$, $\tau > 50$, for maximal velocities $v_{\max} = 1, 2, \ldots, 5$. Note that no averaging over several timesteps or initial conditions has been performed. It is clear that—if the supposition mentioned above were true—averaging over a number of these large τ values and various initial conditions would yield the same result

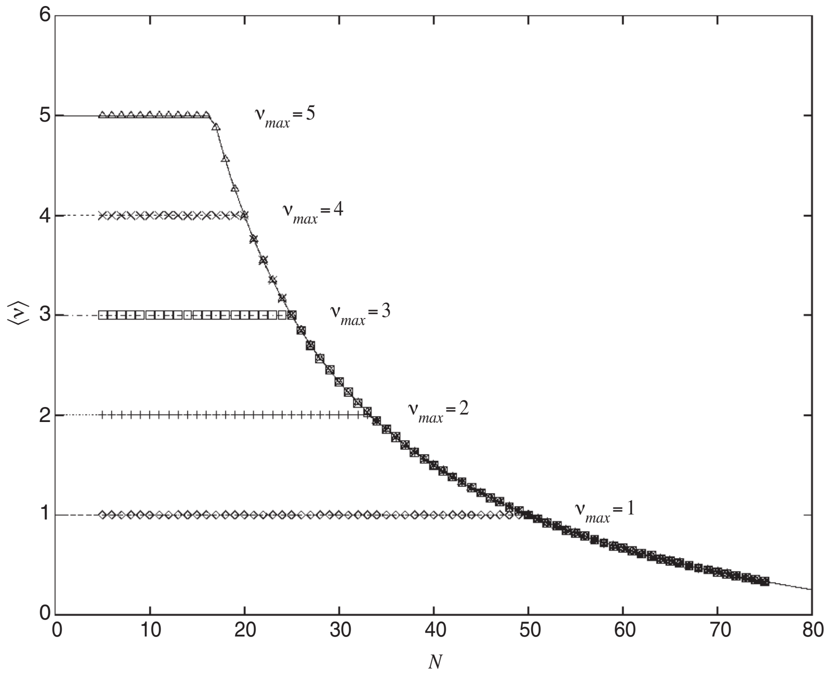


Figure 2. Comparison of the fundamental maxima obtained by simulation (plotted as symbols) with the analytical results of equation (11) (plotted as lines). For each N several initial conditions were chosen at random.

and that therefore the fundamental maxima would correspond exactly to the commonly used fundamental diagram.

However, so far, this is only an assumption, which will be proven step by step in the following sections.

4. An upper bound for the fundamental maxima

In order to obtain an upper bound for the fundamental maxima we consider the first substep in the mathematical representation of the Nagel–Schreckenberg rules in equations (1):

$$\text{Substep 1:}\quad v_{\tau+1}^{(i)} = \min(v_{\max}, v_\tau^{(i)} + 1, \Delta\xi_\tau^{(i)} - 1).$$

By the following transformations we can easily find an upper bound for $\langle v\rangle_{\tau+1}$:

$$\begin{aligned}
\langle v\rangle_{\tau+1} &= \frac{1}{N}\sum_{i=0}^{N-1} v_{\tau+1}^{(i)} && (4)\\
&= \frac{1}{N}\sum_{i=0}^{N-1} \min(v_{\max}, v_\tau^{(i)} + 1, \Delta\xi_\tau^{(i)} - 1) && (5)\\
&\leqslant \frac{1}{N}\sum_{i=0}^{N-1} \min(v_{\max}, \Delta\xi_\tau^{(i)} - 1) && (6)\\
&\leqslant \frac{1}{N}\min\left(\sum_{i=0}^{N-1} v_{\max}, \sum_{i=0}^{N-1}(\Delta\xi_\tau^{(i)} - 1)\right) && (7)\\
&= \frac{1}{N}\min(N v_{\max}, L - N) && (8)\\
&= \min\left(v_{\max}, \frac{L-N}{N}\right). && (9)
\end{aligned}$$

We obtain (6) by neglecting $v_\tau + 1$ in (5) as a possible limiting value, for, in the absence of other limits, this value will increase until the maximum velocity is reached. The states satisfying $\langle v\rangle_\tau = \min(v_{\max}, \frac{L-N}{N})$ will be referred to as maximal states (MSs):

Definition: maximal state. A state $\{\xi_\tau^{(i)}, v_\tau^{(i)}\}$ is called MS, if the corresponding average velocity $\langle v\rangle_\tau$ is

$$\langle v\rangle_\tau := \frac{1}{N}\sum_{i=1}^{N} v_\tau^{(i)} \overset{!}{=} \hat{v}_{N,L,v_{\max}} := \min\left(v_{\max}, \frac{L-N}{N}\right). \tag{10}$$

It is not difficult to construct an MS for each combination of N, L and $v_{\max}$. This means that the set of MSs is not empty. Therefore, the upper bound of the average velocity is also its maximum with respect to all states, i.e. a global maximum. The question remains whether these global maxima are always reached in the course of the system's dynamic development for all initial conditions, that is, whether they are maxima for each trajectory in state space. If in figure 2 one compares the global maxima

$$\hat{v}_{N,L,v_{\max}} := \min\left(v_{\max}, \frac{L-N}{N}\right) \tag{11}$$

(plotted as lines) with the results of the simulations (plotted as symbols), one sees that, indeed, the analytically derived global maxima seem not only to limit the fundamental maxima of the simulations but for each N to be identical for all initial conditions.

This hypothesis is underpinned by the result achieved by Nagel and Herrmann [3] for the special case $v_{\max} = \infty$: $v_{N,L,\infty} = \frac{L-N}{N}$, which agrees perfectly with our result for the upper bound in this special case. In the following section, we will sketch an exact proof for this hypothesis based on concepts and methods of deterministic nonlinear dynamics.

5. Proof of the identity of the global maximum and the fundamental maxima

It was pointed out in section 3.2 that in the simulations after a number of time-steps only fundamental maxima occur as average velocities $\langle v \rangle_\tau$. This suggests that in the course of its dynamical development the system moves to a subset of the MSs and remains there.

This leads us to applying the concepts 'attractive set' and 'attractor', well known in the theory of dynamical systems, to the Nagel–Schreckenberg model. Since in this model all values are discrete the attractor definition, cf [4], has to be adapted accordingly.

5.1. Definitions

In order to indicate the differences between the definitions for the continuum and the adaptations to the discrete case the names are prefixed with a 'd-'.

Definition: d-attractive set. Let $\mathcal{S}$ be a finite discrete state space. A subset $\mathcal{A} \subset \mathcal{S}$ invariant under the dynamics (i.e. $F(\mathcal{A}) = \mathcal{A}$) is called a d-attractive set if there exists a subset $\mathcal{B}_\mathcal{A} \subset \mathcal{S}$ (called the basin) with $\mathcal{A} \subsetneqq \mathcal{B}_\mathcal{A}$ such that for sufficiently large k

$$F^k(\mathcal{B}_\mathcal{A}) \subset \mathcal{A}. \tag{12}$$

Definition: d-topological transitivity. An invariant set $\mathcal{A}$ is called 'd-topological transitive' if

$$\forall\, \Upsilon_1, \Upsilon_2 \in \mathcal{A}: \quad \exists k : F^k(\Upsilon_1) = \Upsilon_2. \tag{13}$$

Definition: d-attractor. A d-attractor is a d-topological transitive d-attractive subset.

5.2. Existence of a d-attractive set of the Nagel–Schreckenberg dynamics

Using the notions introduced we can formulate the conjecture suggested by the simulation results.

Within the set of MSs there exists a d-attractive set of system states.

The proof of this claim will now be sketched in a series of steps; the detailed mathematical proofs can be found in [6].

In order to find d-attractive subsets in the set of MSs, we first have to check for invariance under the dynamics. We are looking for states that themselves are predecessors of MSs. Therefore, we define quite generally:

Definition: maximum-generating state (MGS). A state $X_\tau = \{(\xi_\tau^{(i)}, v_\tau^{(i)})\}$ will be called an MGS if $X_{\tau+1}$ is an MS.

With this definition, we can say that each element of the attractive set must be an MGS. Note that not every MGS is an MS and *vice versa*. In particular, we can easily also construct MSs which are non-maximum-generating. An example is shown in figure 3. Here, at $\tau = 1$ the state is maximal, but not maximum generating: the state at $\tau = 2$ is non-maximal.

$\tau = 0$	`2..5.....5.....`	
$\tau = 1$	`..2.....5.....5`	$\langle \nu \rangle_{\tau=1} = 4$
$\tau = 2$	`.2...3.......5.`	$\langle \nu \rangle_{\tau=2} = \frac{10}{3}$

Figure 3. Time evolution of a small system $L = 15$. The positions at which vehicles are located are labelled with the values of the discrete velocities, empty positions with '.'. Note: not all non-'Garden of Eden' (GoE) MSs are again MGSs. The state at $\tau = 1$ (with $\langle \nu \rangle_{\tau=1} = 4$) is a non-GoE MS (non-GoE, because it has evolved from the state at $\tau = 0$). The successor at $\tau = 2$ yields $\langle \nu \rangle_{\tau=2} = \frac{10}{3}$, and, therefore, is not an MS.

Whether or not an MS is again an MGS is intimately connected with the concept of the so-called 'Garden of Eden states'. The importance of Garden of Eden (GoE) states to mean-field approaches describing the behaviour of stochastic Nagel–Schreckenberg models has already been pointed out by Schadschneider and Schreckenberg [5].

Definition: Garden of Eden state. A system state is called a GoE state of the dynamics if there exists no possible predecessor.

Such a state may therefore occur only as an initial state. The state as $\tau = 0$ in figure 3 is such a GoE state.

The following theorem can be proven:

Theorem. *All non-GoE MGSs (NGoEMGSs) are mapped by the Nagel–Schreckenberg dynamics to non-GoE maximum-generating MSs (NGoEMGMSs). They belong to periodic orbits with period not larger than LN.*

Therefore, the set of NGoEMGMSs automatically constitutes an invariant set. In other words: once the system has reached a (non-GoE) MGS all states following will be MSs. In order to show that the set of these MSs not only is a invariant but also d-attractive, we have to show the existence of a basin of attraction which is mapped into this set by repeated application of the dynamics.

In fact, we have:

Theorem. *All possible states of a Nagel–Schreckenberg system are mapped into the set of NGoEMGMSs by a finite number of iterations.*

This shows that the set of MSs not only contains a d-attractive subset (the NGoEMGMS) but also that this subset is globally d-attractive, i.e. its basin of attraction is the whole state space. Practically this means that the mean speed of the N cars always reaches the fundamental maximum independent of the initial conditions. Averaging over various initial conditions is therefore not necessary in order to obtain the fundamental diagram.

5.3. D-attractors as subsets of the d-attractive set

After having stated the existence of a globally d-attractive subset of the MSs, we will now briefly investigate the d-attractors it is composed of.

If one considers the spacetime diagram of a Nagel–Schreckenberg model, it appears that the motion of the whole system can generally be divided into two parts:

(i) In the initial part a pattern of velocities and relative positions is established by the interactions of the vehicles.
(ii) After that the pattern moves cyclically along the circular lane.

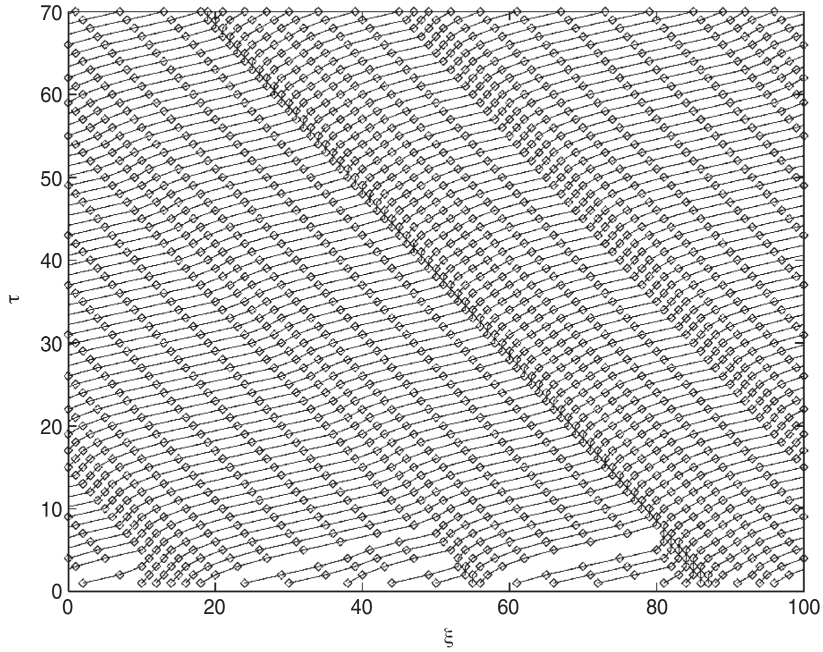


Figure 4. Spacetime diagram of a traffic simulation. In the lower part (early times) one recognizes the formation of a pattern, which, for larger values of τ simply moves cyclically in the system.

A typical spacetime diagram is shown in figure 4. This kind of behaviour suggests identifying the first part with the transient evolution towards an NGoEMGMS and the second part (referred to as 'steady state' by Nagel and Herrmann [3]) with the periodic motion within the set of the maximum-generating MSs. This suggestion can be proven and the following theorem holds:

Theorem. *Given an NGoEMGMS at a time τ, the pattern of velocities and relative positions does not change with time.*

Therefore, two states of the set of the NGoEMGMS with different patterns cannot evolve into each other under the deterministic dynamics. Although both states are elements of the globally d-attractive set, they belong to different d-attractors.

We can summarize the result as follows: starting from a given initial condition the motion of N cars with maximal velocity $v_{\max}$ on a track of length L will evolve into one of, in general, several patterns, all yielding the same mean velocity $\hat{v}_{N,L,v_{\max}}$.

Macroscopically (i.e. by considering only the mean velocity), the different distinct d-attractors cannot be distinguished because they all yield the same mean velocity $\langle v\rangle_\tau$ (fundamental maximum).

If we would take the initial values to be the events of a probability space and the mean velocities $\langle v\rangle_\tau$ as a stochastic process, then the process will be stationary as soon as a d-attractor is reached. In fact, due to the deterministic dynamics the stationary process would become trivial. The Nagel–Schreckenberg model with randomized slowing down is an example of a random dynamical system. The relative influence of the deterministic and the stochastic components in such systems is a difficult subject of its own. Details are given in the monograph by Arnold [7].

Nevertheless, the deterministic model gives some insight into the original randomized version. If the traffic is moving on one of the d-attractors any additional random slowing down of a car will decrease the mean speed of the N cars and move the state off the attractor. The system will be put on a transient. If the probability of such random slowing downs is small there will be enough time for the system to reach an attractor again. The mean speed $\langle \nu \rangle_\tau$ finally reached will be the same since all attractors give the same mean speed. The details of the fluctuations of $\langle \nu \rangle_\tau$ depend on the trajectory structure of the various basins of attraction.

In the next section we will show that the macroscopic equivalence of the various d-attractors is by no means universal and can cease to hold in a slightly changed model such that different attractors can be observed even macroscopically.

6. An extended model

In this section we will show that the concepts and methods described above are not restricted to the original Nagel–Schreckenberg model but also lead to statements about extended models which exhibit a qualitatively different behaviour.

The change consists of a modification of the rules that represents a slow-to-start behaviour. It amounts to assuming that a car, the vehicle preceding which has just started, has to wait for some $\delta_{\max}$ time steps before accelerating itself. Such a behaviour can be achieved by introducing an additional dynamical variable δ_τ which represents the time the vehicle has to remain still. Similar extensions can be found in [8].

6.1. Descriptions of a slow-to-start dynamics

The only difference between the deterministic Nagel–Schreckenberg model and the deterministic slow-to-start Nagel–Schreckenberg model (we will briefly call it the SNS model) consists of an exception in rule (1) and an extension of rule (2). This leads to the following rules:

(i) *Acceleration.* If the velocity ν_τ of a vehicle at time step τ is less than the maximal velocity $\nu_{\max}$ and the distance to the next vehicle ahead is larger than $\nu_\tau + 1$, the speed is advanced by one ($\nu_{\tau+1} = \nu_\tau + 1$). *Exception.* If the vehicle has the velocity $\nu_\tau = 0$ and a non-vanishing remaining waiting time $\delta_\tau > 0$, only this waiting time is decremented by one ($\delta_{\tau+1} = \delta_\tau - 1$).

(ii) *Slowing down due to other cars.* If at time step τ a vehicle at position ξ_τ 'sees' another vehicle at position $\xi_\tau + \Delta\xi_\tau$, it reduces its speed to $\Delta\xi_\tau - 1$ in order to avoid a collision ($\nu_{\tau+1} = \Delta\xi_\tau - 1$).
If this makes the discrete velocity equal to zero $\nu_{\tau+1} = 0$ the waiting time is set to $\delta_{\tau+1} = \delta_{\max}$, where $\delta_{\max}$ is the number of the waiting iterations before the vehicle can start anew.

(iii) *Car motion.* Each vehicle is advanced ν sites ($\xi_{\tau+1} = \xi_\tau + \nu_\tau$).

The original Nagel–Schreckenberg model is just the special case $\delta_{\max} = 0$.
Expressed formally, we find

Substep 1a:

$$\nu^{(i)}_{\tau+1} = \min(\varphi(\nu^{(i)}_\tau, \delta^{(i)}_\tau),\ \Delta\xi^{(i)}_\tau - 1) \tag{14}$$

$$\text{with} \quad \varphi(\nu^{(i)}_\tau, \delta^{(i)}_\tau) = \begin{cases} \min(\nu^{(i)}_\tau + 1, \nu_{\max}) & \text{if} \quad \delta_\tau = 0 \\ 0 & \text{else} \end{cases} \tag{15}$$

Substep 1b:

$$\delta_{\tau+1}^{(i)} = \begin{cases} \delta_{\max} & \text{if} \quad \Delta\xi_\tau^{(i)} - 1 = 0 \\ \max(\delta_\tau - 1, 0) & \text{else} \end{cases} \tag{16}$$

Substep 2:

$$\xi_{\tau+1}^{(i)} = \xi_\tau^{(i)} + v_{\tau+1}^{(i)}. \tag{17}$$

The impact of the change in the dynamics can be clearly recognized by comparing with time evolution series with the same initial conditions but different start delays in figure 5.

```
a)                          b)
0000000..................   0000000..................
000000.1.................   000000.1.................
00000.1..2...............   000000...2...............
0000.1..2...3............   00000.1.....3............
000.1..2...3....4........   00000...2.......4........
00.1..2...3....4.....5...   0000.1.....3.........5...
0.1..2...3....4.....5....   0000...2.......4.........
```

Figure 5. Starting behaviour of a queue of vehicles with (*a*) $\delta_{\max} = 0$ and (*b*) $\delta_{\max} = 1$.

6.2. Fundamental scatter plots of the SNS model

As we did in the case of the original Nagel–Schreckenberg model, we will use the fundamental scatter plot as a starting point of our further investigation, omitting the first 50 steps containing the transients.

In figure 6(*a*) the mean vehicle velocities $\langle v \rangle_\tau$ have been plotted versus the number N of vehicles for $\tau = 51, 52, \ldots, 100$ for a number of simulations with different initial states.

A remarkable qualitative difference between the Nagel–Schreckenberg model and its slow-to-start extension shows up in this figure: a given number N of vehicles may yield different values for $\langle v \rangle_\tau$ even for large τ. The values observed seem to cluster near two distinct curves. The values near the lower curve show a considerable variance while the upper ones seem to follow the curve almost exactly.

Two different effects have to be considered in explaining these results:

(i) Different initial states lead to different d-attractors.
(ii) In the same attractor different values of τ can lead to different values of $\langle v \rangle_\tau$ such that $\langle v \rangle_\tau$ is not constant.

In order to determine which effect causes which change compared with the original Nagel–Schreckenberg model in the fundamental scatter plot we consider

$$\langle\langle v \rangle_\tau\rangle := \lim_{\tau_e \to \infty} \frac{1}{\tau_e - \tau_a} \sum_{\tau=\tau_a}^{\tau_e - 1} \langle v \rangle_\tau. \tag{18}$$

Here τ_a is the time step at which we assume all transients have died out. For $L = 100, v_{\max} = 5$ the value $\tau_a = 50$ empirically has turned out to be suitable as well as approximating the limit by setting $\tau_e = 100$.

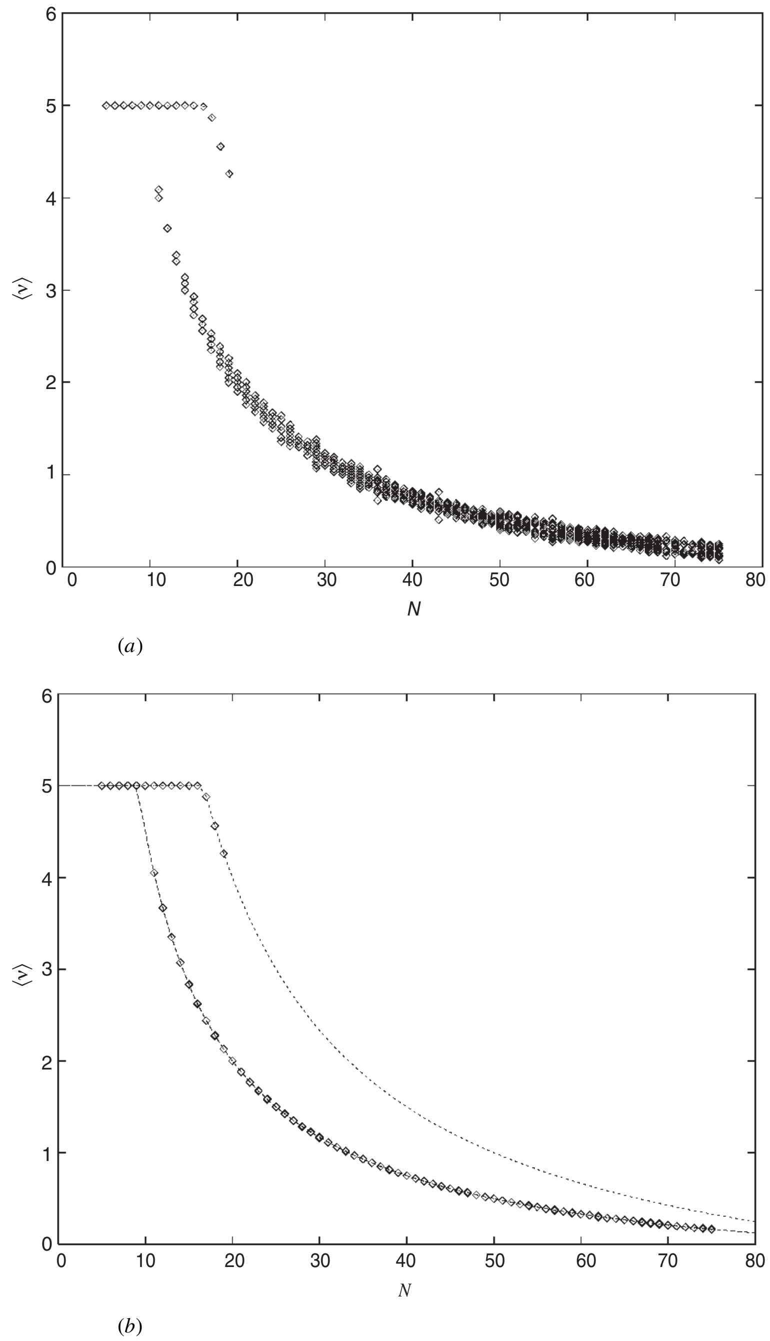


Figure 6. (*a*) Long-term fundamental scatter plot for the SNS model with $\delta_{\max} = 1$. The mean vehicle velocities $\langle v \rangle_\tau$ are plotted versus the number N of vehicles for $\tau = 51, 52, \ldots, 100$ for a number of simulations with different initial states. One recognizes two distinct branches; the values of the lower one show a significant variance. (*b*) Values obtained by averaging over the τ values plotted as points each representing a particular initial condition randomly chosen. The variance observed in (*a*) has vanished. The results obtained for the curves analytically are plotted as dashed curves.

The comparison of the two plots shown in figures 6(a) and (b) allows us to determine the impacts of the two effects mentioned above. The variance of the values near the lower curve in figure 6(a) is due to the variation of velocities within the same attractor (effect 2) since otherwise averaging would have had no effect. However, the existence of two distinct curves must be attributed to coexisting attractors having different values of $\langle\langle v\rangle_\tau\rangle$ (effect 1) since time averaging does not make any of the curves disappear.

To each of the branches belongs a specific basin of attraction. If one were to compute the fundamental diagram in the usual sense one would have to average over all initial conditions considered possible. The two branches could no longer be resolved. For each value of N the two branches would be replaced by the common average, each branch being weighted by the size of its basin of attraction.

The curve with the higher values of $\langle v\rangle_\tau$ corresponds exactly to that obtained analytically in section 4 for the Nagel–Schreckenberg model without the slow-to-start rule (plotted with dots in figure 6(b)). The case where $\langle v\rangle_\tau$ lies on the upper curve in the following will be denoted by an index ($\uparrow$), the other one by ($\downarrow$). If one considers two spacetime diagrams for the same N which belong to the cases ($\uparrow$) and ($\downarrow$), respectively (figure 7), one finds an important difference in the long-term behaviour: in the ($\uparrow$) case no car ever stops while for ($\downarrow$) at each time step $\tau > \tau_a$ at least one vehicle i has the velocity $v_\tau^{(i)} = 0$.

6.3. D-attractive sets and d-attractors in the SNS model

As in the ordinary Nagel–Schreckenberg model the existence of a d-attractive set and its constituting attractors can be shown. Considering the two cases ($\uparrow$) and ($\downarrow$) separately, we have:

Case ($\uparrow$)*: dynamical transition to Nagel–Schreckenberg behaviour.* It can be shown that all non-GoE states with no car velocity $v^{(i)} = 0$ tend to the same attractors which they would approach in the Nagel–Schreckenberg model without the slow-to-start rule. These 'non-GoE non-zero' states (NGoENZSs) and all states which lead to them in course of time evolution form the basin of the attractors. For these attractors the relation

$$\langle v\rangle_\tau = \hat{v}^{(\uparrow)}_{N,L,v_{\max}} := \min\left(v_{\max}, \frac{L-N}{N}\right), \tag{19}$$

which we have derived in section 4, holds.

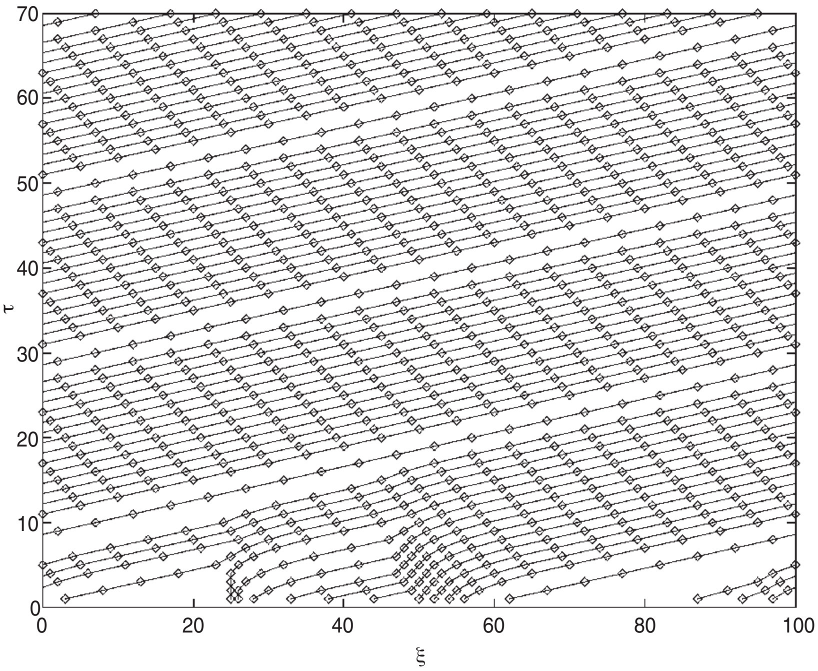
Case ($\downarrow$)*: stable cyclical moving jams.* For the complement of the set of NGoENZS states and their predecessors, i.e. the set of all states that never lead to an NGoENZS, it can be shown that their time evolution guarantees an 'eternal' jam that moves cyclically in a direction opposite to the movement of the cars.

From the number of time steps it takes a vehicle in the front position of an eternal jam to move on the cyclic lane until it attains the front position of the same jam again, the relation

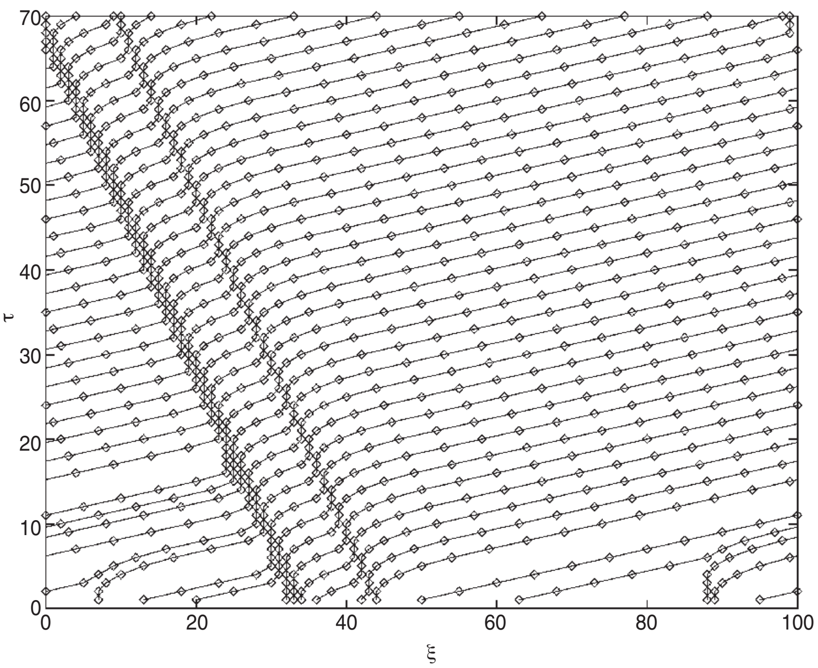
$$\langle\langle v\rangle_\tau\rangle = \hat{v}^{(\downarrow)}_{N,L,v_{\max}} := \frac{L-N}{2N} \tag{20}$$

can be derived.

The comparison of the values $\langle\langle v\rangle_\tau\rangle$ obtained by simulation with the theoretical results in figure 6(b) shows excellent agreement.



(a)



(b)

Figure 7. Spacetime diagrams (a) for the case (↑) and (b) for the case (↓).

7. Conclusion

It has been shown that an upper bound for the long-term fundamental diagram of the Nagel–Schreckenberg model in the deterministic case can be easily derived starting from a closed mathematical representation. Furthermore, it has been shown using concepts and methods of nonlinear dynamics that this upper bound is always reached independent of the initial conditions. The states of the long-term behaviour form a globally d-attractive subset which can be characterized as 'non-Garden of Eden maximum-generating MSs'. These d-attractive sets are composed of several d-attractors all having the same macroscopic properties. The inclusion of slow-to-start effects in the dynamics shows that this macroscopic equivalence critically depends on the particulars of the model. In the extended model considered one finds coexisting attractors having two different average velocities.

This paper might serve as a starting point for further investigations. On the theoretical side investigating the interplay of stochastic influences and the deterministic skeleton of the model could give further understanding of the way traffic patterns form and disappear. On the experimental side, the discovery of intrinsic attractor-like structures in real-world traffic (by measurements and nonlinear time series analysis) could indicate the applicability of new efficient methods for traffic control on the basis of nonlinear dynamics. Such methods might shift the system state in a well controlled manner from 'slow' to 'fast' attractors. Experimental results presented by Kerner in [9] are particularly interesting in this context.

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